A Discrete Model for the Dynamics of Sandpile Surfaces

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Abstract

A new Cellular Automata model for the dynamics of sandpile surfaces is presented in this work. Following the BCRE model (Bouchaud et al., 1994) main ideas, we propose the use of two variables, instead of only the one that has been used in Cellular Automata models so far. The model predicts sandpile properties observed in other models, and in real systems, such as slope roughness decay or uphill propagation of avalanches. In addition, the introduction of the second variable allows the prediction of characteristics, such as the appearance of a second critical angle, that have been predicted by continuous models. The main advantage of the proposed model is that its formulation is simpler and easier to interpret than that of the continuous models. Also, the resulting numerical scheme is simpler and more efficient, allowing for a wider range of applications, including interactive simulation.

Introduction

Granular systems behaviour has been widely studied during the last decades, due to its many applications to industry. Many works have been devoted to study the dynamics of sandpile surfaces (Bouchaud et al., 1994; Bouchaud et al., 1995; Aradian et al., 1999; Prigozhin and Zaltzman, 2001; Prigozhin and Zaltzman, 2003), specially in the particular case when the mean slope of the surface is close to the so called *angle of repose* of the system.

In the last decade, the increase in computer capabilities has propitiated that sandpile models are used in more complex simulations, with applications to film industry or virtual reality simulation (Müller et al., 2003). However, classical models are computationally expensive, and provide an inadequate description of the system given that its graphical representation is necessary. Moreover, real-time or interactive simulation of such systems is far from being possible.

In this work we present a new Cellular Automata model that has been developed within a series of efforts aimed to obtain models that allow real-time, interactive, simulation of granular systems, for virtual reality applications.

In the next section the reader is briefly introduced in the field of the dynamics of sandpile surfaces. Following, the CA model is formally described and explained. Then the dynamic properties of the model are reviewed, and compared to those observed in empirical studies and in the continuous model, as studied by Bouchaud et. al. in (Bouchaud et al., 1994). Finally the results of this work are discussed.

Dynamics of Sandpile Surfaces

Schematically, sandpile surface dynamics can be described with very few ideas. Sand grains can be piled up until the surface of the heap is higher than the *angle of repose* of the system. When this happens, an avalanche occurs and some grains roll down the slope, until the slope of the surface is again reduced below the *angle of repose*.

This basic idea was used by Bak et. al. (Bak et al., 1988) to define the update rule of a Cellular Automata, in order to illustrate the dynamics of self-organised critical systems. Later on, other authors have proposed continuous models aimed to describe the dynamics of sandpile surfaces which reflect the same basic principles. Hwa and Kardar (Hwa and Kardar, 1989; Hwa and Kardar, 1992) proposed a partial differential equation for the dynamics of the height of the surface.

Models based on this basic principle describe the system using a unique variable, the height of the system at each point, and reproduce the appearance of the critical angle and also predict a smoothing of the surface of the system, reaching an almost homogeneous slope along the system.

Later Bouchaud et. al. (Bouchaud et al., 1994; Bouchaud et al., 1995) modified that model in order to introduce two variables, instead of just the height of the system. Their model considers two layers; a static layer, of resting material grains, and a rolling layer that contains the grains that are falling down the slope. They considered the interchange of matter among the two layers, and also added a convection term to the dynamics of the rolling layer. The main contribution of the work by Bouchaud et al. was the prediction of a second critical angle. If the mean slope is above this angle any perturbation leads to a catastrophic avalanche.

This work has become a referent in this field, and has motivated several other works, aimed both to the refinement of their model (Aradian et al., 1999; Hadeler and Kuttler, 1999; Prigozhin and Zaltzman, 2001; Prigozhin and Zaltzman, 2003) and to its application to industry problems (Hadeler and Kuttler, 2001).

However, the use of continuous models (based on Partial Differential Equations (PDEs)) for the description of the dynamics of sandpile surfaces involves many technical considerations from the computational point of view.

The numerical schemes used in the solution of PDEs try to give a very precise solution to the equations. But, in contrast with the high precision that is sought with usual PDE methods, all the models mentioned above are phenomenological models; this means that the inner interaction between the particles of the system are nor considered individually.

In those works, the analysis of the results focus on qualitative aspects, and only medium to large scale behaviour of the system is considered. For this reason, the degree of precision that can be achieved with the solution of a set of PDEs is unnecessary. In other words, the effort to implement a PDEs method and its computational cost is somehow too hard for the analysis that will be done on the data.

The Cellular Automata approach proposed by Bak et. al. (Bak et al., 1988), which is computationally simpler and closer to the motivating ideas of the model, has not played a relevant role in the study of sandpile surface dynamics. Instead, this approach has been used in works that study statistical properties of granular systems (Prado and Olami, 1992; Nerone and Gabbanelli, 2001; Chen and Nijs, 2002), and as a paradigmatic approach to critical systems modelling.

More recently, Pla-Castells et. al. (Pla-Castells, 2003; Pla-Castells et al., 2004) have proposed a variation of Cellular Automata models in order to achieve a better description of the surface dynamics using Cellular Automata. The main motivation of these works was to obtain a computational model that can be used for real-time interactive simulation of granular systems, in applications such as driving simulators or virtual reality environments. With the requirement of real-time in mind, Cellular Automata models have proven a very reliable approach to the problem, that highly overcomes the performance of continuous models, and of Discrete Event Modelling (Müller et al., 2003).

In this work we go further and show that Cellular Automata models can provide the same predictions as continuous models do, by means of simpler and more efficient models. We analyse the qualitative behaviour of a two variable Cellular Automata model in order to show that its dynamics has the main properties predicted by the well known BCRE model. Specially, it is shown that the introduction of a second variable in the Cellular Automata model leads to the same behaviour as that observed in the continuous model.

PROPOSED MODEL

The model is described following the notation by Bouchaud et. al. (Bouchaud et al., 1994). For simplicity, it is presented for a one-dimensional Cellular Automata, for which two variables are considered at each cell; the height of the standing layer $h(i,t)$ above the centre of cell i, and the height of the rolling layer $R(i,t)$, at time t. Two parameters are considered for the dynamics of the rolling layer; the drift velocity of the rolling grains v, and a diffusion constant D. A parameter γ controls the exchange of matter among the two layers, together with the *angle of repose*, S_c.

Also, some additional parameters related to the Cellular Automata description of the system are considered. N is the number of cells of the system and d is a cell's length; the system will have a length of $N \times d$ length units. The neighbourhood of cell i (the set of cells that directly influence its evolution) is taken as the set of cells $\mathbf{V} = \{(i-1),(i+1)\}$.

Formal Cellular Automata Model

The Cellular Automata is described as

$$CA = \langle \mathbf{M}, \mathbf{V}, \mathbf{S}, \boldsymbol{\varphi} \rangle \quad (1)$$

where

- $\mathbf{M} = \{(i) \in \mathbb{N}, 1 \leq i \leq N\}$ is the set of cells that form the Cellular Automata.
- $\mathbf{V}$ is the neighbourhood of a cell, as described above.
- $\mathbf{S} = S_h \times S_R$ is the set of possible states for each cell:

  $S_h = \mathbb{R}$ is the height of the static layer;
  $S_R = \mathbb{R}$ is the height of the rolling layer.
- $\boldsymbol{\varphi} : \mathbf{S}^3 \rightarrow \mathbf{S}$ is the transition map of the Automata.

The Transition Map

Given a cell, the transition map, $\boldsymbol{\varphi} : \mathbf{S}^3 \rightarrow \mathbf{S}$, takes the value of its current state (the value of its variables) and that of its neighbours, and determines its new value after a time stepthe new state the state of a cell after a time step. The transition map is computed as follows.

For each cell $(i) \in \mathbf{M}$, let $h \in S_h$, the height of the static layer and $R \in S_R$ the height of the rolling layer above the cell. The gradient of each layer is computed, using the difference of height with the neighbour cells:

$$dh(i) = \frac{h(i+1) - h(i)}{d} \quad (2)$$

$$dR(i) = \frac{R(i+1) - R(i)}{d} \quad (3)$$

Then, each cell is updated as follows

$$\begin{aligned} R(i) &\leftarrow R(i) + (-vR(i) + DdR(i)) \\ &\quad -R(i)\gamma(dh(i) - |S_c|) \quad (4) \\ h(i) &\leftarrow h(i) + R(i)\gamma(dh(i) - |S_c|) \quad (5) \\ R(i+1) &\leftarrow R(i+1) - (-vR(i) + DdR(i)) \quad (6) \end{aligned}$$

ANALYSIS OF THE MODEL

The introduction of a second variable in the model leads, in the continuous case, to the existence of a second critical angle S_d, for the static layer (Bouchaud et al., 1994). If the mean slope of the static phase S_0 is above this second critical angle, $S_0 < S_d$, any avalanche leads to a catastrophic phenomenon, in which the static layer looses all of its matter while the rolling layer grows along the whole slope.

This behaviour can also be observed in the proposed Cellular Automata model; with the introduction of a second variable in the model, a second critical value is observed in the slope of the system. The sandpile surface dynamics, in this case, is similar to that observed in the continuous model. Figure 1 shows the results of a simulation that illustrate this behaviour.

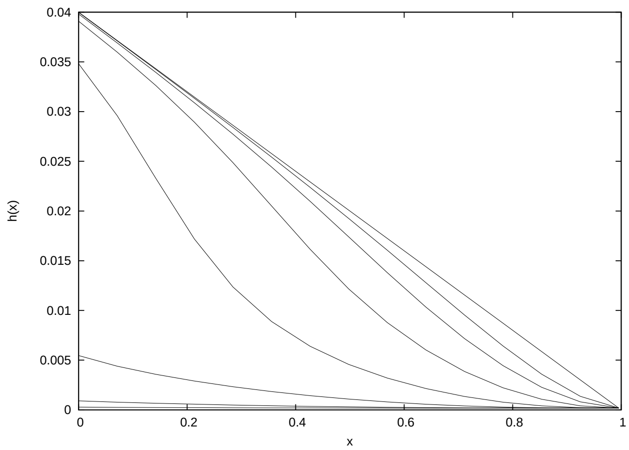

Figure 1: Evolution along time of a perturbation of the rolling layer at the bottom of the slope. The different lines indicate the profile of the static layer at successive instants of time. In this case, it can bee seen that the thickness of the static layer decreases until it is almost zero. The parameters of the model were $v = 0.1$, $D = 0.1$ and $\gamma = 0.01$

Also, the dynamics of the single-variable models can still be observed; the new model is an extension of previous ones. The numerical experiments performed so far reveal the following relevant properties.

1. If a system with slope close to the *angle of repose* is perturbed locally, then an avalanche occurs. The result is that the surface returns to an angle below the critical angle (see Figure 2).

2. Surface roughness tends to dissipate, due to the diffusion terms in the model. This makes that a perturbation spreads down the slope instead of just travelling across it.

3. Propagation of avalanches happens not only downwards, but also upwards, as predicted by continuous models and observed in real granular systems (Bouchaud, 1998; Daerr and Douady, 1999; Aranson and Tsimring, 2001) (see Figure 3).

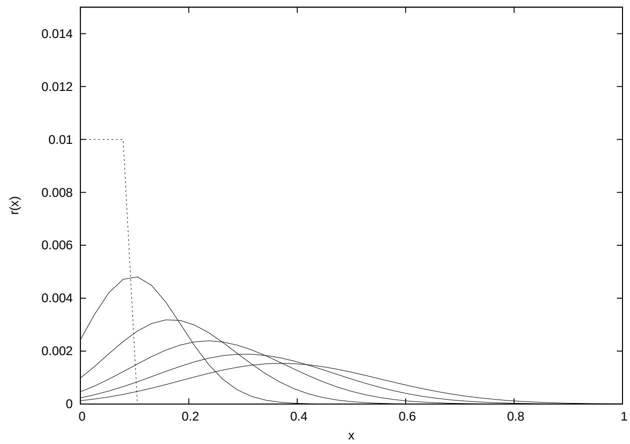

Figure 2: Evolution along time of a perturbation of the rolling layer on top of the slope. The dashed line indicates the initial profile of the rolling layer, while the different continuous lines indicate its successive distributions. The static layer has a slope to the right, slightly lower than the *repose angle*. The parameters of the model were $v = 2.5$, $D = 0.1$ and $\gamma = 0.01$.

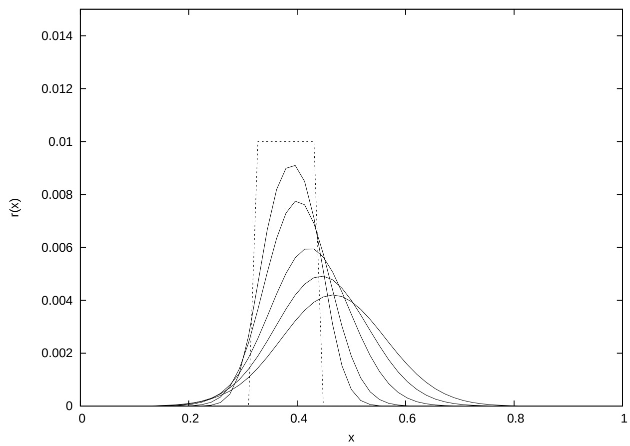

Figure 3: Evolution along time of a perturbation of the rolling layer in the middle of the slope. Again, the dashed line indicates the initial profile of the rolling layer, while the different continuous lines indicate its successive distributions. The region affected by the avalanche includes part of the cells above the initial perturbation. The parameters of the model were the same as in Figure 2.

Thus, the new model provides a proper description of the dynamics of sandpile surfaces, with the additional advantages observed in Cellular Automata models:

1. Cellular Automata models are discrete in space, thus the computation is not performed on a discretization of the data, but on the data itself.

2. Due to the statistical properties of the dynamics of granular systems, and more precisely, to the fact that they often reflect self-organised criticality, the implementation of two-dimensional systems allows an exhaustive optimisation of the model. This leads to a computational cost that is linear respect to the side of the system, instead of quadratic.

3. The discrete description of the system is also very adequate for the graphical representation of a granular system. This makes Cellular Automata models more suitable for its use in computer graphics applications (such as virtual reality or simulation for training).

Part of the validation is still ongoing, and some characteristics observed in sandpile surfaces, such as the statistical properties of the model, have not been investigated in depth yet. However, preliminary simulations, together with previous results obtained with similar models (Pla-Castells, 2003) give a good level of confidence that they will be met.

CONCLUSIONS

A Cellular Automata model for the dynamics of sandpile surfaces has been presented in this work. Previous Cellular Automata models already were able to reflect the main characteristics observed in the dynamics of sandpile surfaces, such as statistical properties of avalanches, slope roughness decay or uphill propagation of avalanches.

The new model extends such properties, by means of the introduction of a second variable; both the static and the rolling layer are considered, as in the well known BCRE model. The introduction of two variables, which is new to Cellular Automata description of sandpiles, allows to predict a second critical angle, as continuous models and experimental results did.

The success in the prediction of such property of the dynamics of sandpile surfaces, shows that proper models can be obtained without the need of PDEs, just using a simple model based upon a local phenomenological description of the dynamics. This description is simpler to understand and to implement than PDEs, and it avoids unnecessary technical considerations needed for the numerical solution of differential equations.

Although a lot of work is still to be done, the preliminary results of this work show that a reliable and efficient family of models can be available for the description of the dynamics of granular systems' surface. Together with other works, already presented by the authors in the field of interactive models of granular systems (Pla-Castells et al., 2006), the final results of this ongoing research open the possibility to develop interactive simulators involving granular systems in the short term.

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